# THE ARROW–PRATT INDEXES OF RISK AVERSION AND CONVEX RISK MEASURES THEY IMPLY

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ABSTRACT. The Arrow–Pratt index of relative risk aversion combines the important economic concepts of elasticity and marginal utility. The index is used by many authors writing in relation to utility theory. Kenneth Joseph Arrow, the Stanford economics professor emeritus, Nobel laureate in economics in 1972, and member of the American Academy of Sciences, was the co-creator, along with the late Gerard Debreu, himself a Nobel laureate in 1983, fellow member of the American Academy, and recipient of the French Legion of Honor.

## 1. INTRODUCTION

The Arrow–Pratt index of relative risk aversion $R_u(c)$ (with the subscript and argument referring to utility and consumption functions, respectively,) combines the important economic concepts of elasticity and marginal utility. There is also an Arrow–Pratt index of absolute risk aversion $r_u(c)$, *v.i.*, which gives rise to a convex risk measure. The relative index is used by many authors writing in relation to utility theory, *e.g.*, Karatzas and Shreve (Karatzas and Shreve 1998, p. 179, p. 188). Kenneth Joseph Arrow, the Stanford economics professor emeritus, Nobel laureate in economics in 1972, and member of the American Academy of Sciences, was the co-creator. Along with the late Gerard Debreu, himself a Nobel laureate in 1983, fellow member of the American Academy, and recipient of the French Legion of Honor, "Arrow produced the first rigorous proof of the existence of a market-clearing equilibrium, given certain restrictive assumptions," [Wikipedia]. See their seminal paper on this subject (Arrow and Debreu 1954). The absolute risk index is here developed as a basis for a convex risk measure.

Specifically, $R_u(c)$ is the negative of the elasticity of marginal utility, where typically $u(c)$ is a utility function of a rate of consumption. The independent variable could also be a function of total consumption, or of wealth, or of another variable, and frequently is stochastic. Marginal utility is simply $u'(c)$. In practical terms one thinks of the index as the fractional change in this marginal utility resulting from a fractional change in the independent variable, inverting the sign. Equivalently, the index is the negative of the ratio of the logarithmic differentials of marginal utility to the independent variable.

The negative sign is included so that the index will be positive in its ordinary ranges. In this mode a higher index value is associated with a greater fractional change in marginal utility for a given fractional change in the independent variable. The higher index value

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identifies an individual of greater risk aversion, one who chooses to avoid situations providing the greater fractional loss in utility.

In general the elasticity of any function $y(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is

$$\operatorname{Elas}[y(x)] := \frac{d\log y}{d\log x} = \frac{dy}{y}\Big/\frac{dx}{x} = \frac{x}{y}\frac{dy}{dx},$$

So

$$R_u(c) = -\operatorname{Elas}[u'(c)] = -\frac{d\log u'(c)}{d\log c} = -\frac{du'(c)}{u'(c)}\Big/\frac{dc}{c} = -\frac{c}{u'(c)}\frac{du'(c)}{dc} = -\frac{c\,u''(c)}{u'(c)},$$

which is in the form the index is usually presented. Observe that the index is a dimensionless quantity, so that changes in the unit of measurement of either the independent variable or of the utility have no effect.

The index of absolute risk aversion $r_u(c)$ is similar, but is stated with reference to the simple differential of the independent variable, rather than the logarithmic differential, and as such is not an elasticity.

$$r_u(c) = -\frac{d\log u'(c)}{dc} = -\frac{1}{u'(c)}\frac{du'(c)}{dc} = -\frac{u''(c)}{u'(c)} = -\log' u'(c) \tag{1.1}$$

This index is not dimensionless, but rather has the reciprocal dimensions of the independent variable. Thus, if the independent variable is rate of consumption, expressed, *e.g.*, as Euros per day, then $r_u(c)$ has units of days per Euro. In the above cited reference the index also appears in this form (Karatzas and Shreve 1998, Section 4.7, Example 7.8, pp. 194–196).

Typically in an optimization problem an agent endeavors to maximize the expected discounted utility from consumption

$$E\left[\int_0^T \exp\left(-\int_0^t \beta(s)\,ds\right) u\big(c(t)\big)\,dt\right] \tag{1.2}$$

over the life of a process assuming an interest function $\beta(s)$, which along with $c(t)$ may be either deterministic or stochastic. (The utility function $u(\cdot)$ is usually assumed deterministic.) See (Karatzas and Shreve 1998, Section 4.2, p. 162).

## 2. Cumulative risk indexes with deterministic consumption

For a primer on coherent and convex risk measures see Appendix A.

Define now a new absolute index, the *absolute consumptive intensity*, as the reciprocal of $r_u(c)$, thus inverting the units, *e.g.*, to Euros per day. This quantity can be integrated to produce a *cumulative absolute consumptive intensity* over a period of time $[0, T]$. In this regard, let

$$\widetilde{r}_u(c) := \frac{1}{r_u(c)}, \tag{2.1}$$

and referring to Equation (1.1), while recognizing explicitly the rate of consumption dependent on time $c(t)$, and assuming appropriate discounting as in Equation (1.2), let the [ordinary] variable

$$S_r(T) := -\int_0^T \exp\left(-\int_0^t \beta(s)\,ds\right)\widetilde{r}_u\big(c(t)\big)\,dt \tag{2.2}$$

So

$$S_r(T) \;= +\int_0^T \exp\left(-\int_0^t \beta(s)\,ds\right)\frac{1}{\log' u'\big(c(t)\big)}\,dt \tag{2.3}$$

After the integration on time, the units of $S_r(T)$ are: *money*, *e.g.*, Euros.

An example serves to illustrate.

**Example 2.1.** *Let* $u(c) = \log c$, *let* $c(t) = 1/t$, *and let* $\beta(s) = \rho > 0$, *a constant. Then*

$$\begin{aligned} S_r(T) &= \int_0^T e^{-\rho t} t\,dt \\ &= -\left(\frac{t}{\rho} + \frac{1}{\rho^2}\right) e^{-\rho t}\Big|_0^T \\ &= -\left(\frac{T}{\rho} + \frac{1}{\rho^2}\right) e^{-\rho T} + \frac{1}{\rho^2} \end{aligned}$$

*Scaling* $S_r(T)$ *by the factor* $\rho^2$ *provides the probability distribution function*

$$\Phi_r(T) = 1 - (1 + \rho T)e^{-\rho T}$$

*For the choices* $\rho = 0.05$ *and* $T \in [0, 200]$ *see Figure 1 for a plot of* $\Phi_r(T)$. *The limits for all derivatives at zero and infinity are zero. The inflection point is at* $T = 1/\rho$. *The range of the function is* $[0, 1)$.

Return now to the relative index $R_u(c)$ and make the parallel analysis. Define a new relative index, the *relative consumptive intensity*, as the reciprocal of $R_u(c)$, also dimensionless. This quantity can be integrated to produce a *cumulative relative consumptive intensity* over a period of time $[0, T]$.

$$\widetilde{R}_u(c) := \frac{1}{R_u(c)} \tag{2.4}$$

Let the variable

$$S_R(T) := -\int_0^T \exp\left(-\int_0^t \beta(s)\,ds\right)\widetilde{R}_u\big(c(t)\big)\,dt \tag{2.5}$$

So

$$= +\int_0^T \exp\left(-\int_0^t \beta(s)\,ds\right)\frac{1}{c(t)\log' u'\big(c(t)\big)}\,dt \tag{2.6}$$

After the integration on time, the units of $S_R(T)$ are: *time*.

An example serves to illustrate.

**Example 2.2.** *Let* $u(c) = \log c$*, let* $c(t) = 1/t$*, and let* $\beta(s) = \rho > 0$*, a constant. Then*

$$S_R(T) = \int_0^T e^{-\rho t} t^2 \, dt$$
$$= -\left(\frac{t^2}{\rho} + \frac{2t}{\rho^2} + \frac{2}{\rho^3}\right) e^{-\rho t} \Big|_0^T$$
$$= -\left(\frac{T^2}{\rho} + \frac{2T}{\rho^2} + \frac{2}{\rho^3}\right) e^{-\rho T} + \frac{2}{\rho^3}$$

*Scaling* $S_R(T)$ *by the factor* $\rho^3/2$ *provides the probability distribution function*

$$\Phi_R(T) = 1 - \frac{1}{2}\left(2 + 2\rho T + \rho^2 T^2\right) e^{-\rho T}$$

*For the choices* $\rho = 0.05$ *and* $T \in [0, 200]$ *see Figure 2 for a plot of* $\Phi_R(T)$*. The limits for all derivatives at zero and infinity are zero. The inflection point is at* $T = (1/\rho)\left[\left(1 + \sqrt{(5)}\right)/2\right]$*, the second factor being the 'golden ratio.' The range of the function is* $[0, 1)$*.*

## 3. Induced risk measures

Proceeding, I now demonstrate that $S_r(T)$, as defined through Equations (2.1) and (2.2), and realized by Equation (2.3), induces a convex risk measure. The intuition is that the inverse of the logarithm, a concave function, is the exponential, a convex function. One needs only typical conditions on $u(c)$ and mild conditions on $c(t)$. A similar development obtains for $S_R(T)$, with reference to Equations (2.4), (2.5), and (2.6).

## LIST OF FIGURES

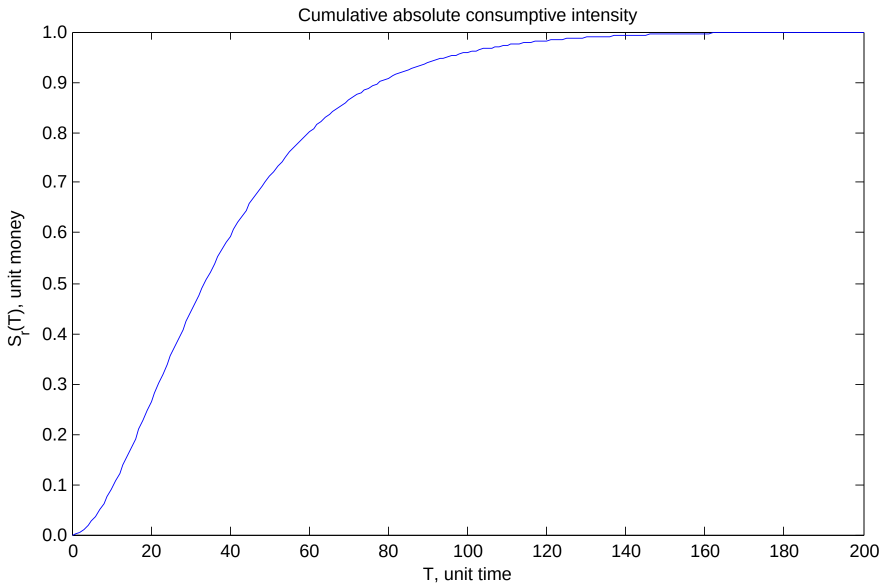


FIGURE 1. Cumulative absolute consumptive intensity

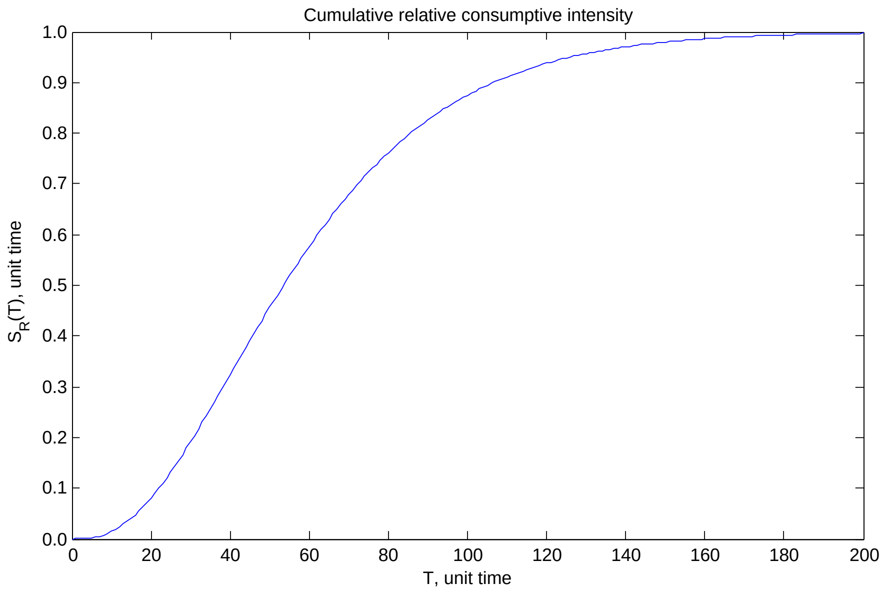


FIGURE 2. Cumulative relative consumptive intensity

## APPENDIX A. COHERENT AND CONVEX RISK MEASURES

The author gratefully acknowledges this source for the content of this Appendix (Ta Thi Kieu 2008).

Make the following assumptions.

- $(\Omega, \mathcal{F}, P)$ is a general probability space.
- $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ is the net worth of a financial position at terminal time T.
- $\mathcal{X}$ is the space of all financial positions $\{X\}$; $\mathcal{X} = L^p(\Omega, \mathcal{F}, P)$, with $1 \leqslant p \leqslant \infty$.

### A.1. **Static risk measures.**

**Definition A.1.** A function $\rho : \mathcal{X} \to \mathbb{R}$ is a *static risk measure* (or simply a *risk measure*) if, for $X, Y \in \mathcal{X}$, the following six conditions hold.

(1) *convexity:* $\rho\big(\alpha X + (1-\alpha)Y\big) \leqslant \alpha\rho(X) + (1-\alpha)\rho(Y), \quad \alpha \in (0,1),\ \forall X, Y \in \mathcal{X}$
(2) *positivity:* $X \geqslant 0 \Longrightarrow \rho(X) \leqslant \rho(0)$
(3) *constancy:* $\rho(\alpha) = -\alpha, \quad \forall \alpha \in \mathbb{R}$
(4) *translability:* $\rho(X+\beta) = \rho(X) - \beta, \quad \forall \beta \in \mathbb{R},\ \forall X \in \mathcal{X}$. In particular, $\rho\big(X + \rho(X)\big) = \rho(X) - \rho(X) = 0$
(5) *sublinearity:*
  (a) *positive homogeneity:* $\rho(\alpha X) = \alpha\rho(X), \quad \forall \alpha \geqslant 0,\ \forall X \in \mathcal{X}$
  (b) *subadditivity:* $\rho(X+Y) \leqslant \rho(X) + \rho(Y), \quad \forall X, Y \in \mathcal{X}$
(6) *lower semi-continuity:* $\{X \in \mathcal{X} \mid \rho(X) \leqslant \gamma\}$ is closed in $\mathcal{X}$ for any $\gamma \in \mathbb{R}$

**Definition A.2.**

(1) A risk measure is *coherent* if it satisfies 2, 4, and 5.
(2) A risk measure is *convex* if it satisfies 1 and 6, and if $\rho(0) = 0$.
(3) A risk measure is *acceptable/unacceptable* as $\rho(X) \lessgtr 0$.

*Remark.* The value $\mp\rho(X)$ is the maximal/minimal amount that an investor can withdraw/deposit from/to the position X without making it unacceptable/acceptable. For guidance on coherent and convex risk measures, respectively, see (Artzner, Delbaen, Eber, and Heath 1999) and (Frittelli and Rosazza Gianin 2002).

The following two representation theorems are important to the sequel. See (Rosazza Gianin 2006, Theorem 3, p. 21, and Theorem 4, p. 22) for the general context. The authors are cited below for the proofs.

**Theorem A.3** (Coherent risk measure representation)**.**

$$\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$$

*is a coherent risk measure satisfying lower semi-continuity if and only if there exists a non-empty closed convex set $\mathcal{P}$ of* P*-absolutely continuous probability measures such that*

$$\rho(X) = \sup_{Q \in \mathcal{P}} E_Q[-X], \quad \forall X \in L^\infty$$

*Proof.* Omitted. See (Delbaen 2002).

**Theorem A.4** (Convex risk measure representation)**.**

$$\rho : \mathcal{X} \to \mathbb{R}$$

*is a convex risk measure satisfying positivity and translability if and only if there exists a convex functional* $F : \mathcal{X}' \to \mathbb{R} \cup \{\infty\}$ *satisfying* $\inf_{x' \in \mathcal{X}'} F(x') = 0$ *such that*

$$\rho(X) = \sup_{Q \in \mathcal{P}} \left\{ E_Q[-X] - F\left(\frac{dQ}{dP}\right) \right\}, \quad \forall X \in \mathcal{X},$$

*where*

$$\mathcal{P} := \left\{ Q \ll P \,\middle|\, \frac{dQ}{dP} \in \mathcal{X}' \text{ and } F\left(\frac{dQ}{dP}\right) < \infty \right\}$$

*is a non-empty convex set.*

*Proof.* Omitted. See (Frittelli and Rosazza Gianin 2002).

A.2. **Dynamic risk measures.** In a seminal paper Emanuela Rosazza Gianin axiomatically introduced the concept of a dynamic risk measure, stating, "that almost any dynamic coherent or convex risk measure comes from a conditional g-expectation ..." (Rosazza Gianin 2006, Section 1, p. 20). That one phrase inspired two definitions, and forms the basis of this investigation.

**Definition A.5.** A function $\rho_t : \mathcal{X} \to \mathbb{R},\ t \in [0, T]$, is a *dynamic risk measure* if, for $X, Y \in \mathcal{X}$, the following six conditions hold.

(1) *convexity:* $\rho_t\big(\alpha X + (1-\alpha)Y\big) \leqslant \alpha\rho_t(X) + (1-\alpha)\rho_t(Y), \quad \alpha \in (0,1),\ \forall X, Y \in \mathcal{X}$
(2) *positivity:* $X \geqslant 0 \Longrightarrow \rho_t(X) \leqslant \rho_t(0)$
(3) *constancy:* $\rho_t(\alpha) = -\alpha, \quad \forall \alpha \in \mathbb{R}$
(4) *translability:* $\rho_t(X + \beta) = \rho_t(X) - \beta, \quad \forall \beta \in \mathbb{R},\ \forall X \in \mathcal{X}$. In particular, $\rho_t\big(X + \rho_t(X)\big) = \rho_t(X) - \rho_t(X) = 0$
(5) *sublinearity:*
  (a) *positive homogeneity:* $\rho_t(\alpha X) = \alpha\rho_t(X), \quad \forall \alpha \geqslant 0,\ \forall X \in \mathcal{X}$
  (b) *subadditivity:* $\rho_t(X + Y) \leqslant \rho_t(X) + \rho_t(Y), \quad \forall X, Y \in \mathcal{X}$
(6) *lower semi-continuity:* $\{X \in \mathcal{X} \mid \rho_t(X) \leqslant \gamma\}$ is closed in $\mathcal{X}$ for any $\gamma \in \mathbb{R}$

**Definition A.6.**

(1) A dynamic risk measure is *coherent* if it satisfies 2, 4, and 5.
(2) A dynamic risk measure is *convex* if it satisfies 1 and 6, and if $\rho_t(0) = 0$.
(3) A dynamic risk measure is *acceptable/unacceptable* as $\rho_t(X) \lesseqgtr 0$.

*Remark.* The value $\mp\rho_t(X)$ is the maximal/minimal amount that an investor can withdraw/deposit from/to the position $X$ at time $t \in [0, T]$ without making it unacceptable/acceptable.

A.3. **Unconditional and conditional g-expectations.** Consider a functional

$$\begin{aligned} g\colon\ & \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \\ & (\omega, t, y, z) \mapsto g(\omega, t, y, z) \end{aligned}$$

(sometimes written without the first argument when a path $\omega$ is assumed) with the 'usual' assumptions and axioms as elucidated in (Rosazza Gianin 2006, Subsection 1.2, pp. 22–23). As well, consider the following backward stochastic differential equation (BSDE), which has a unique solution, stated and cited in the same reference.

$$\begin{aligned} -dY_t &= g(t, Y_t, Z_t)\,dt - Z_t\,dB_t, \quad \forall t \in [0,T] \\ Y_T &= X \end{aligned} \tag{A.1}$$

Then one has

**Definition A.7.**

(1) the $g$-*expectation* $\mathrm{E}_g[X] := Y_0^X$
(2) the *conditional* $g$-*expectation* $\mathrm{E}[X \mid \mathcal{F}_t] := Y_t^X$

One proceeds to redefine static risk measures via $g$-expectations, and dynamic risk measures via conditional $g$-expectations.

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